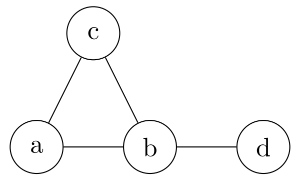
Kaleidoscope Mathematics – Part 2 Exam Solutions

1. Consider random walk $X_0, X_1, \dots$ on the graph:



Define the function f on the vertices of the graph by

$$f(\mathrm{a}) = 0.1, \qquad f(\mathrm{b}) = -2, \qquad f(\mathrm{c}) = 8, \qquad f(\mathrm{d}) = 0.$$

Find $\mathbb{E}_{\mathrm{a}}[f(X_2)]$, that is, the expected value of $f(X_2)$ for the walk started from $X_0 = \mathrm{a}$.

Solution. With the labeling

$$\mathrm{a} \leftrightarrow 1,\ \mathrm{b} \leftrightarrow 2,\ \mathrm{c} \leftrightarrow 3,\ \mathrm{d} \leftrightarrow 4,$$

the transition matrix is

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 1/3 & 0 & 1/3 & 1/3 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Then, $\mathbb{E}_{\mathrm{a}}[f(X_2)]$ is the entry in the first row of $P^2 f$. We compute

$$P^2 f = \begin{pmatrix} 5/12 & 1/4 & 1/6 & 1/6 \\ 1/6 & 2/3 & 1/6 & 0 \\ 1/6 & 1/4 & 5/12 & 1/6 \\ 1/3 & 0 & 1/3 & 1/3 \end{pmatrix} \begin{pmatrix} 0.1 \\ -2 \\ 8 \\ 0 \end{pmatrix} = \begin{pmatrix} 7/8 \\ 1/60 \\ 57/20 \\ 27/10 \end{pmatrix}$$

(in fact, to save time, we could have computed only the first row of P^2). Hence, $\mathbb{E}_{\mathrm{a}}[f(X_2)] = (P^2 f)(\mathrm{a}) = \frac{7}{8}$.

2. Explain why all states in an irreducible Markov chain have the same period.

Solution. As seen in class, if two states are accessible from each other, then they have the same period. In an irreducible chains, all states are accessible from each other. Hence they all have the same period.

3. Prove that, for any two states x and y of a Markov chain, we either have $[x] = [y]$ or $[x] \cap [y] = \varnothing$. You may use the fact that, for any three states a, b, c, if we have $a \rightsquigarrow b$ and $b \rightsquigarrow c$, then $a \rightsquigarrow c$.

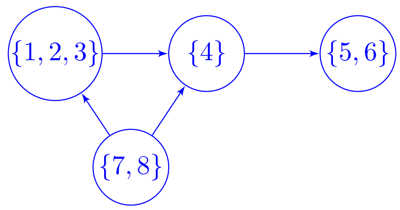
Solution. Assume that $[x] \cap [y] \neq \varnothing$, and let us show the it necessarily follows that $[x] = [y]$. Fix an element $z \in [x] \cap [y]$ (it exists by the assumption). We then have $x \rightsquigarrow z$ and $y \rightsquigarrow z$. We then have that $x \leftrightsquigarrow y$. Now, fix an arbitrary state $w \in [x]$. Since $x \leftrightsquigarrow w$ and $y \leftrightsquigarrow x$, we have $y \leftrightsquigarrow w$, so $w \in [y]$. This shows that $[x] \subset [y]$. Arguing in the same way we also get $[y] \subset [x]$, so $[x] = [y]$.

4. A Markov chain has states space $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and transition matrix P given below (the entry $P(i, j)$ denotes the probability of jumping from i to j, for all $i, j \in S$):

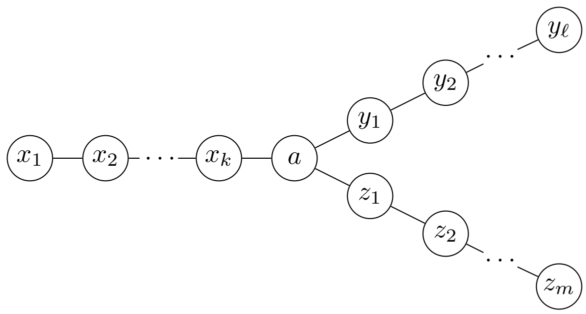
$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.4 & 0.2 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 & 0 & 0 \\ 0.4 & 0 & 0 & 0 & 0 & 0 & 0 & 0.6 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0.6 & 0 \end{pmatrix}$$

Identify all communicating classes of this chain and give their representation with a directed graph (as done in the lectures: each vertex of the directed graph should represent one communicating class of the chain, and a directed edge from one vertex to another should indicate that it is possible to go from one communicating class to the other with a single jump). Classify all communicating classes as recurrent or transient.

Solution. The communicating classes are: $\{1,2,3\}$, $\{4\}$, $\{7,8\}$, $\{5,6\}$; the first three are transient and the last one is recurrent. The directed graph representing the communicating classes is as follows:



5,6. Consider random walk on the graph represented below (where k, ℓ, m) are positive integers.



(a) Find $\mathbb{P}_a(H_{y_\ell} < H_{z_m})$ (that is, the probability that the walk started from a reaches y_ℓ before reaching z_m). Your answer may depend on k, ℓ, m.

(b) Find $\mathbb{P}_a(H_{\{y_\ell, z_m\}} < H_{x_1})$ (that is, the probability that the walk started from a reaches either y_ℓ or z_m before reaching x_1). Your answer may depend on k, ℓ, m.

Solution.

(a) Define, for each vertex v of the graph,

$$f(v) := \mathbb{P}_v(H_{y_\ell} < H_{z_m}).$$

Note that

$$f(x_1) = f(x_2), \quad f(x_2) = \frac{1}{2} \cdot f(x_1) + \frac{1}{2} \cdot f(x_3), \quad \ldots, \quad f(x_k) = \frac{1}{2} \cdot f(x_{k-1}) + \frac{1}{2} \cdot f(a).$$

As in the gambler's ruin problem, we obtain

$$f(x_3) - f(x_2) = f(x_2) - f(x_1), \quad f(x_4) - f(x_3) = f(x_3) - f(x_2), \quad \ldots, \quad f(a) - f(x_k) = f(x_k) - f(x_{k-1}),$$

which now gives (since $f(x_1) = f(x_2)$):

$$f(x_1) = f(x_2) = \cdots = f(x_k) = f(a).$$

We then write

$$f(a) = \frac{1}{3} \cdot f(x_k) + \frac{1}{3} \cdot f(y_1) + \frac{1}{3} \cdot f(z_1) \quad \Longrightarrow \quad \frac{2}{3} \cdot f(a) = \frac{1}{3} \cdot f(y_1) + \frac{1}{3} \cdot f(z_1) \quad \Longrightarrow \quad f(a) = \frac{1}{2} \cdot f(y_1) + \frac{1}{2} \cdot f(z_1).$$

We also have the equations

$$f(y_\ell) = 1, \quad f(y_{\ell-1}) = \frac{1}{2} \cdot f(y_{\ell-2}) + \frac{1}{2} \cdot f(y_\ell), \quad f(y_1) = \frac{1}{2} \cdot f(y_2) + \frac{1}{2} \cdot f(a)$$

and

$$f(z_m) = 0, \quad f(z_{m-1}) = \frac{1}{2} \cdot f(z_{m-2}) + \frac{1}{2} \cdot f(z_m), \quad f(z_1) = \frac{1}{2} \cdot f(z_2) + \frac{1}{2} \cdot f(a).$$

Hence, the equations for f in the line segment connecting y_ℓ and z_m are precisely the same as for the original gambler's ruin problem in this line segment. The solution is then given by

$$f(a) = \frac{m}{m+\ell}.$$

(b) For each vertex v of the graph we define

$$g(v) := \mathbb{P}_v(H_{\{y_\ell, z_m\}} < H_{x_1}).$$

We have

$$g(x_1) = 0, \quad g(y_\ell) = g(z_m) = 1.$$

Moreover, we have

$$g(x_2) = \frac{1}{2} \cdot g(x_1) + \frac{1}{2} \cdot g(x_3), \quad \ldots, \quad g(x_{k-1}) = \frac{1}{2} \cdot g(x_{k-2}) + \frac{1}{2} \cdot g(x_k), \quad g(x_k) = \frac{1}{2} \cdot g(x_{k-1}) + \frac{1}{2} \cdot g(a).$$

As in the gambler's ruin problem, we have

$$g(x_3) - g(x_2) = g(x_2) - g(x_1), \quad g(x_4) - g(x_3) = g(x_3) - g(x_2), \quad \ldots, \quad g(a) - g(x_k) = g(x_k) - g(x_{k-1}),$$

so, letting $\delta := g(x_2) = g(x_2) - g(x_1)$, we obtain

$$g(x_2) = \delta, \quad g(x_3) = 2\delta, \quad \ldots, \quad g(x_k) = (k-1)\delta, \quad g(a) = k\delta.$$

Defining

$$\omega := g(y_\ell) - g(y_{\ell-1}) = 1 - g(y_{\ell-1}) \quad \text{and} \quad \sigma := g(z_m) - g(z_{m-1}) = 1 - g(z_{m-1})$$

and arguing similarly, we obtain

$$g(y_{\ell-1}) = 1 - \omega, \quad g(y_{\ell-2}) = 1 - 2\omega, \quad \ldots, \quad g(y_1) = 1 - (\ell-1)\omega, \quad g(a) = 1 - \ell\omega$$

and

$$g(z_{m-1}) = 1 - \sigma, \quad g(z_{m-2}) = 1 - 2\sigma, \quad \ldots, \quad g(z_1) = 1 - (m-1)\sigma, \quad g(a) = 1 - m\sigma.$$

We hence have

$$g(a) = k\delta = 1 - \ell\omega = 1 - m\sigma \quad \Longrightarrow \quad \omega = \frac{1-k\delta}{\ell}, \ \sigma = \frac{1-k\delta}{m}.$$

Finally, we write the equation

$$\begin{aligned} k\delta = g(a) = \frac{1}{3} \cdot g(x_k) + \frac{1}{3} \cdot g(y_1) + \frac{1}{3} \cdot g(z_1) &= \frac{1}{3} \cdot (k-1)\delta + \frac{1}{3} \cdot (1 - (\ell-1)\omega) + \frac{1}{3} \cdot (1 - (m-1)\sigma) \\ &= \frac{1}{3} \cdot (k-1)\delta + \frac{1}{3} \cdot \left(1 - (\ell-1)\frac{1-k\delta}{\ell}\right) + \frac{1}{3} \cdot \left(1 - (m-1)\frac{1-k\delta}{m}\right). \end{aligned}$$

Isolating δ and simplifying, this gives

$$\delta = \frac{\frac{1}{\ell} + \frac{1}{m}}{1 + \left(\frac{1}{\ell} + \frac{1}{m}\right)k}, \qquad \text{so } g(a) = k\delta = \frac{\frac{1}{\ell} + \frac{1}{m}}{\frac{1}{k} + \frac{1}{\ell} + \frac{1}{m}}.$$